There is, finally,

$$R = A\left\{\frac{\sin a'}{r'}\left(\frac{\cos^2 a'}{r'} - \frac{\cos^2 a}{r}\right) - \frac{4}{5}\frac{\cos a'}{r'}\left(\frac{\sin a' \cos a'}{r'} - \frac{\sin a \cos a}{r}\right)\right\}$$

and

$$W = A\left\{\cos a'\left(\frac{\cos^2 a'}{r'} - \frac{\cos^2 a}{r}\right) + \frac{4}{5}\sin a'\left(\frac{\sin a' \cos a'}{r'} - \frac{\sin a \cos a}{r}\right)\right\}.$$

The distortion of the helix will change the total angle φ, turned by the helix into $\varphi + d\varphi$; where φ is $2\pi n$; and the height of the helix from h to $h + dh$. Since $\varphi = (s \cos a)/r$ then $\varphi + d\varphi = (s \cos a')/r$ and $h + dh = s \sin a'$. Taking $d\varphi$ and dh as infinitesimal we have $s \cos a' = s \cos a - \tan a dh$. Since

$$\frac{1}{r} = \frac{\varphi}{\sqrt{s^2 - h^2}},$$

then

$$\frac{1}{r^2} = \frac{\varphi + d\varphi}{\sqrt{s^2 - (h + dh)^2}}.$$

After proper reductions and substitutions

$$R = \frac{A}{s^3}\left(\frac{4s^2 + h^2}{5(s^2 - h^2)}\varphi^2 dh - \frac{\varphi h}{5} d\varphi\right) \quad \text{and} \quad W = \frac{A}{S^3}\left(-\frac{\varphi h}{5} dh + \frac{5s^2 - h^2}{5} d\varphi\right).$$

There are two possible cases: the terminal is held fast against twist and hence $d\varphi = 0$; and the terminals are free to turn and the resultant moment W is zero. The former is the usual case in practice and will be the only one discussed here. Either condition, with R given, renders the problem a determinate one. Introducing the former condition, then

$$R = \frac{A}{5s^2}\,\frac{4s^2 + h^2}{s^2 - h^2}\varphi^2 dh \quad \text{and} \quad W = -\frac{A\varphi h}{5S^3} dh.$$

If p is the radius of the wire and S is the unit shear, we have, since the section is a circle,

$$S = pW/I_s.$$

The resilience must equal the work done, whence Res. $= \frac{1}{2} Rdh$.

With the conditions as given in the problem, dh is first to be found, and noting that $s^2 - h^2 = \varphi r^2$, there is

$$dh = \frac{5s^3 r^2 R}{A(4s^2 + h^2)}.$$

Further, since h^2 is usually infinitesimal in comparison with $4s^2$, the formula may be simply written

$$dh = \frac{5sr^2 R}{4A}.$$

This is the expression used in the numerical substitution below.

Note, that within reasonable limits of the pitch, s is practically constant so that the shortening of the spring is independent of the pitch.

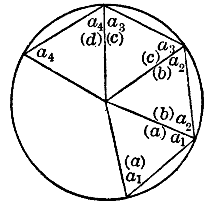
With the data given in the problem, $r = 1.5''$; $p = 0.129''$; $R = 50$ lbs. and E is assumed as 30,000,000 lbs. so that $A = 6{,}570$, $n = 20$. The horizontal projection of the length of the spring is $2\pi nr$, which is equal to 188.5″, so, that for small pitches, s may be taken about 190. The shortening is then 4.1″. Substituting this in the expression for W and then introducing this value of W for finding S, the maximum stress, *i. e.*, shear, is 855 lbs. per sq. in. The resilience is found to be 102.5 in. lbs.

329. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

A smooth circular table is surrounded by a smooth circular rim. Show that the ball, whose coefficient of restitution is e projected along the table from a point in the rim making an angle $\tan^{-1}e^{3/2}$ with the radius through the point, will return to the point of projection after three rebounds.

I. Solution by G. Paaswell, N. Y. City.

Before attacking this special problem let us establish the general case of k points of contact, *i. e.*, of $k-1$ rebounds. Denote the reciprocal of e by r and $\tan a$ by s (referring to the figure). From the laws of elastic impact, $\tan b = r \tan a$; then $\tan c = r^2 \tan a$; $\tan d = r^3 \tan a$ and so forth. From the figure, $2a + 2b + 2c + \cdots = (k-2)\pi$; whence



$$\tan^{-1} s + \tan^{-1} rs + \tan^{-1} r^2 s + \cdots = \frac{\pi}{2}(k-2)$$

It may be shown by developing a few cases and then by induction that

$$\tan^{-1} a_1 + \tan^{-1} a_2 + \tan^{-1} a_3 + \tan^{-1} a_4 + \cdots + \tan^{-1} a_k$$

$$= \tan^{-1}\left(\frac{{}_1\Sigma_k - {}_3\Sigma_k + {}_5\Sigma_k \cdots \pm {}_k\Sigma_k}{1 - {}_2\Sigma_k + {}_4\Sigma_k \cdots \pm {}_{k-1}\Sigma_k}\right) = \frac{\pi}{2}(k-2),$$

when k is odd and

$$= \tan^{-1}\left(\frac{{}_1\Sigma_k - {}_3\Sigma_k + {}_5\Sigma_k \cdots \pm {}_{k-1}\Sigma_k}{1 - {}_2\Sigma_k + {}_4\Sigma_k \cdots \pm {}_k\Sigma_k}\right) = \frac{\pi}{2}(k-2),$$

when k is even.

${}_p\Sigma_k$ represents the sum of the k arguments taken p at a time. Taking the tangent of both members $\left(\text{the second member is } (k-2)\frac{\pi}{2}\right)$ and noting that when k is odd, $\tan(k-2)\frac{\pi}{2}$ is infinite and hence the denominator must vanish, and similarly, when k is even, the numerator of the corresponding fraction must vanish, we have

$${}_1\Sigma_k - {}_3\Sigma_k + {}_5\Sigma_k \pm \cdots\cdots {}_{k-1}\Sigma_k = 0 \text{ (k is even)},$$

$$1 - {}_2\Sigma_k + {}_4\Sigma_k \pm \cdots\cdots {}_{k-1}\Sigma_k = 0 \text{ (k is odd)}.$$

When the sequence of a's forms a geometric progression it may be shown without difficulty that

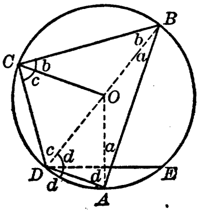
$${}_p\Sigma_k = r^{p(p-1)/2} s^p \frac{1 - r^{-A_p}}{1-r}; \qquad A_p = \frac{k!}{(k-p)!p!};$$

and the condition equations become

$$s\frac{1-r^{A_1}}{1-r} - r^3 s^3 \frac{1-r^{A_3}}{1-r} + r^{10} s^5 \frac{1-r^{A_5}}{1-r} \cdots \pm r^{(k-1)(k-2)/2} s^{k-1} \frac{1-r^k}{1-r} = 0, \ k, \text{ even};$$

$$1 - rs^2\frac{1-r^{A_2}}{1-r} + r^6 s^4 \frac{1-r^{A_4}}{1-r} \cdots \pm r^{(k-1)(k-2)/2} s^{k-1} \frac{1-r^k}{1-r} = 0, \ k, \text{ odd}.$$

These are the general equations for any number of rebounds. Thus, $k = 2$ (one rebound) $s(1+r) = 0$, or $s = 0$, $a = 0$; $k = 3$ (two rebounds), $1 - rs^2\dfrac{1-r^3}{1-r} = 0$, $s^2 = \dfrac{1}{r(1+r+r^2)}$ $a = \tan^{-1}\sqrt{\dfrac{e^3}{1+e+e^2}}$; $k = 4$ (3 rebounds, the given case), $s\dfrac{1-r^4}{1-r} - r^3 s^3 \dfrac{1-r^4}{1-r} = 0$; $1 - s^2 r^3 = 0$, $s = \dfrac{1}{r^{3/2}}$; whence $a = \tan^{-1} e^{3/2}$.



II. Solution by Charles A. Hutchinson, Wittenberg College, Springfield, Ohio.

Let A be the point of projection, and B, C, D, the points of first, second and third rebounds, respectively.

Assume that after the third rebound, the ball does not pass along DA, but along DE.

Then $\tan a = \sqrt{e^3}$, $\tan b = 1/e \cdot \tan a = \sqrt{e}$, $\tan c = 1/e \cdot \tan b = 1/\sqrt{e}$, and $\tan f = 1/e \cdot \tan c = 1/\sqrt{e^3} = \cot a$. Hence, $f = 90° - a$.

Since $\tan c = 1/\sqrt{e} = \cot b$, therefore, $c = 90° - b$. Hence, $a + b + c + f = 180$. But $a + b + c + d = 180°$. Hence, $d = f$.

Hence, DA and DE coincide and the ball does return to A.

Also solved by HORACE OLSON, A. M. HARDING, and HAROLD T. DAVIS.

NUMBER THEORY.

230. Proposed by E. B. ESCOTT, Ann Arbor, Michigan.

Find three numbers such that their sum, the sum of their squares, and the sum of their cubes shall be a cube.

SOLUTION BY E. T. BELL, Seattle, Washington.

Presumably the proposer wished three *integers*, (x, y, z) satisfying the conditions. No restrictions being imposed, we may assume one of the cubes to be zero, and choose $(x + y + z) = 0$, which simplifies the problem very considerably, reducing it in effect to the solution in integers (m, v) of

(1) $$m^2 = 36v^3 + 1.$$

A solution (x, y, z) is:

$$\begin{array}{llll}
x = 146, & x^2 = 21\,316, & x^3 = 3\,112\,136, \\
y = -1\,314, & y^2 = 1\,726\,596, & y^3 = -2\,268\,747\,144, \\
z = 1\,168, & z^2 = 1\,364\,224, & z^3 = 1\,593\,413\,632, \\
w = -876, & & w^3 = -672\,221\,376,
\end{array}$$

whence

$$x + y + z = 0^3,$$
$$x^2 + y^2 + z^2 = x^3,$$
$$x^3 + y^3 + z^3 = w^3 = (-6x)^3,$$

which may be most readily checked from the resolutions into prime factors:

$$x = 2 \times 73; \quad y = -2 \times 3^2 \times 73; \quad z = 2^4 \times 73; \quad w = -2^2 \times 3 \times 73;$$

clearly, (xt^3, yt^3, zt^3), where t is an arbitrary integer, is also an integral solution.

More generally, (m, v) being any solution of (1), and k arbitrary,

(2) $$\begin{aligned}
x &= 18(m^2 + 3)(m + 1)^2(m - 1)^2k^3, \\
y &= -9(m^2 + 3)(m + 1)^3(m - 1)^2k^3, \\
z &= 9(m^2 + 3)(m + 1)^2(m - 1)^3k^3,
\end{aligned}$$

give

(3) $$\begin{aligned}
x + y + z &= 0^3, \\
x^2 + y^2 + z^2 &= \{2^3 3^4 k^2 v^4 (m^2 + 3)\}^3 = \{2^5 3^4 v^4 k^2 (3^2 v^2 + 1)\}^3, \\
x^3 + y^3 + z^3 &= \{-2^5 3^7 (m^2 + 3) v^7 k^3\}^3 = \{-2^7 3^7 v^7 k^3 (3^2 v^2 + 1)\}^3,
\end{aligned}$$

which, for $k = 2^{-4}3^{-2}$, and the solution $(m, v) = (17, 2)$ of (1) gives the solution first stated.

Note.—The analysis for solution 2, which would take about two pages of the MONTHLY, can be written out if thought of sufficient interest, but it involves nothing new. Incidentally, the above solution carries with it that of many other curious indeterminate systems, when (3) is combined with Newton's formulæ for the sums of like powers of the roots of an algebraic equation; thus, we are shown how to find integers (a, b, c, p, r) satisfying

$$a^5 + b^5 + c^5 = 30p^3,$$
$$a^7 + b^7 + c^7 = 28r^3.$$

There is an infinite chain of such results, which are all more or less in the nature of accidents.